An improved Altman type generalization of the Brézis–Browder ordering principle

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Abstract. *By using a modified argument, we prove an improvement of our former Altman type generalization of the Brézis–Browder ordering principle which yields a stronger maximum principle.*

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Introduction

In 1976, to unify a number of diverse results in nonlinear functional analysis, H. Brézis and F. E. Browder [3] proved the following general ordering principle.

Theorem 1. *Let* X *be an ordered set; for* $x \in X$ *denote* $S(x) = \{ y \in X;\ y \geq x \}$. *Let* $\phi : X \to \mathbb{R}$ *be a function satisfying*

(1) $x \leq y$ *implies* $\phi(x) \leq \phi(y)$ *;*

(2) for any increasing sequence $\{x_n\}$ *in* X *such that* $\phi(x_n) \leq C < \infty$ *for all* n*, there exists some* $y \in X$ *such that* $x_n \leq y$ *for all* n *;*

(3) for every $x \in X$ *there exists* $u \in X$ *such that* $x \leq u$ *and* $\phi(x) < \phi(u)$ *.*

Then, for each $x \in X$, $\phi\big(S(x)\big)$ *is unbounded.*

As a direct consequence of this theorem, the above authors derived the following maximum principle.

Corollary 1. *Let* $\phi X \to \mathbb{R}$ *be a function, bounded above, and satisfying*

(1') $x \leq y$ *and* $x \neq y$ *imply* $\phi(x) < \phi(y)$ *;*

(4) for any increasing sequence $\{x_n\}$ *in* X*, there exists some* $y \in X$ *such that* $x_n \leq y$ *for all* n *.*

Then, for each $a \in X$*, there exists some* $\bar{a} \in X$ *such that* $a \leq \bar{a}$ *and* $\bar{a}$ *is maximal (i. e.,* $S(\bar{a}) = \{\bar{a}\}$ *).*

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The importance of this corollary lies mainly in the fact that it easily yields a simplified version of Ekeland's variational principle and hence also of Caristi's fixed point theorem. Moreover, it can also be used to prove Danes' drop theorem [3].

In 1982, having in mind the function Φ, defined by $\Phi(x, y) = \phi(x) - \phi(y)$ for all $x, y \in X$, M. Altman [1] generalized the above theorem in the following less satisfactory form.

Theorem 2. *Let $(X \ \leq)$ be an ordered set such that every totally ordered sequence $\{x_n\} \subset X$ such that $x_{n+1} \leq x_n$ for $n = 1, 2, \ldots$ has a minorant, i. e., there exists an element $y \in X$ such that*

(i) $y \leq x_n$ *for* $n = 1, 2, \ldots$.

Let $w = \Phi(x, y)$ be a real-valued function defined for all $x, y \in X$ such that for each given y, $\Phi(\cdot, y)$ is bounded from below on $S(y) = [z \in X \mid z \leq y]$;

(ii) $\Phi(x, y) \leq 0$ *if* $x \leq y$ *for all* $x, y \in X$;

(iii) Φ *is non-increasing in the second variable, i. e., for any given* $x \in X$, $\Phi(x, y_2) \leq \Phi(x, y_1)$ *if* $y_1 \leq y_2$ *for all* $y_1, y_2 \in X$;

(iv) $\liminf \Phi(x_{n+1}, x_n) = 0$.

Then for each $x \in X$ there exists a $y \in X$ such that $y \leq x$ and $z \leq y$ implies $\Phi(z, y) = 0$.

As a direct consequence of this *Theorem 2*, the above author derived the following

Corollary 2. *Suppose that the hypotheses of* Theorem 2 *are satisfied with the assumption (ii) replaced by the stronger one*

(iib) $x \leq y$ *and* $x \neq y$ *imply* $\Phi(x, y) < 0$.

Then for each $x \in X$ there exists $\bar{x} \in X$ such that $\bar{x} \leq x$ and $\bar{x}$ is minimal, i. e., $z \leq \bar{x}$ implies $z = \bar{x}$.

In 1984, M. Turinici [19] gave a better formulation and a metric generalization of the above theorem which also yields a maximum principle. Altman's theorem, in a somewhat improved form, has also been included in Zeidler [23, p. 515].

In 2001, not being aware of the works of M. Turinici, the present author also proved a generalization of Altman's theorem and derived a maximum principle. However, it has turned out that this theorem also contained several superfluous hypotheses.

Therefore, in the present paper we shall show that, by using a somewhat modified argument, we can actually prove a stronger result which may have a wider range of applications. For this, it is convenient to introduce some particular terminology.

1. Some general definitions

Definition 1. *If X is a set, then a function Φ of X^2 into $\overline{\mathbb{R}}$ will be called an écart on X.*

Example 1. *If φ and ψ are functions of X into $\mathbb{R}$, then the function Φ, defined by $\Phi(x, y) = \varphi(y) - \psi(x)$ for all $x, y \in X$, is a natural écart on X.*

Definition 2. *A set X equipped with a relation $\leq$ will be called a goset (generalized ordered set).*

Remark 1. *A goset X will be called reflexive, symmetric and transitive, if the relation in it has the corresponding property.*

Definition 3. *If Φ is an écart on a goset X, then the function γ_Φ, defined by*

$$\gamma_\Phi(x) = \sup_{y \geq x} \Phi(x, y)$$

for all $x \in X$, will be called the gauge of Φ.

Remark 2. *Note that if X is a reflexive goset and Φ is as in* Example 1, *then $-\infty < \gamma_\Phi(x)$ for all $x \in X$. Moreover, if $a \in X$ is such that φ is bounded above on $[a, +\infty[= \{x \in X : \ a \leq x\}$, then $\gamma_\Phi(a) < +\infty$.*

Concerning the function γ_Φ, it is also worth noticing the following

Proposition 1. *If Φ is an écart on a goset X such that for any $x_1, x_2, y \in X$, with $x_1 \leq x_2$ and $x_2 \leq y$, there exists $z \in X$, with $x_1 \leq z$, such that $\Phi(x_2, y) \leq \Phi(x_1, z)$, then γ_Φ is decreasing.*

Proof. Suppose that $x_1, x_2 \in X$ such that $x_1 \leq x_2$. If $x_2 \not\leq y$ for all $y \in X$, then because of $\sup(\emptyset) = -\infty$ we have $\gamma_\Phi(x_2) = -\infty$. Therefore, $\gamma_\Phi(x_2) \leq \gamma_\Phi(x_1)$ automatically holds.

If $y \in X$ such that $x_2 \leq y$, then by the assumption of the theorem there exists $z \in X$, with $x_1 \leq z$ such that $\Phi(x_2, y) \leq \Phi(x_1, z)$. Hence, by the definition of the supremum, it is clear that

$$\Phi(x_2, y) \leq \Phi(x_1, z) \leq \sup_{w \geq x_1} \Phi(x_1, w) = \gamma_\Phi(x_1).$$

Therefore, by the definition of the supremum, $\gamma_\Phi(x_2) = \sup_{y \geq x_2} \Phi(x_2, y) \leq \gamma_\Phi(x_1)$ also holds. □

Now, as an immediate consequence of the above proposition, we can also state

Corollary 3. *If Φ is an écart on a transitive goset X such that for any $x_1, x_2, y \in X$, with $x_1 \leq x_2$ and $x_2 \leq y$, we have $\Phi(x_2, y) \leq \Phi(x_1, y)$, then γ_Φ is decreasing.*

Remark 3. *Note that if X is a transitive goset and Φ is as in* Example 1 *such that ψ is increasing, then γ_Φ is already decreasing by the above corollary.*

2. A generalized ordering principle

The importance of the above observations on γ_Φ lies mainly in the following

Lemma 1. *If Φ is an écart on a goset X such that*

(1) γ_Φ is decreasing;

(2) $-\infty < \gamma_\Phi(x)$ for all $x \in X$;

(3) $\gamma_{\Phi}(a) < +\infty$ *for some* $a \in X$ *;*

then there exists an increasing sequence $(x_n)_{n=1}^{\infty}$ *in* X*, with* $x_1 = a$*, such that*

$$\lim_{n\to\infty} \gamma_{\Phi}(x_n) = \lim_{n\to\infty} \Phi(x_n, x_{n+1}).$$

Proof. Define $x_1 = a$. Then, by (2) and (3), we have $-\infty < \gamma_{\Phi}(x_1) < +\infty$. Therefore,

$$\gamma_{\Phi}(x_1) - 1 < \gamma_{\Phi}(x_1) = \sup_{y \ge x_1} \Phi(x_1, y).$$

Hence, by the definition of the supremum, it is clear that there exists $x_2 \in X$, with $x_1 \le x_2$, such that

$$\gamma_{\Phi}(x_1) - 1 < \Phi(x_1, x_2).$$

Moreover, by using (2) and (1), we can also note that $-\infty < \gamma_{\Phi}(x_2) \le \gamma_{\Phi}(x_1) < +\infty$. Therefore,

$$\gamma_{\Phi}(x_2) - 2^{-1} < \gamma_{\Phi}(x_2) = \sup_{y \ge x_2} \Phi(x_2, y).$$

Hence, by the definition of the supremum, it is clear that there exists $x_3 \in X$, with $x_2 \le x_3$, such that

$$\gamma_{\Phi}(x_2) - 2^{-1} < \Phi(x_2, x_3).$$

Moreover, by using (2) and (1), we can note that $-\infty < \gamma_{\Phi}(x_3) \le \gamma_{\Phi}(x_2) < +\infty$.

Now, by induction, it is clear that there exists an increasing sequence $(x_n)_{n=1}^{\infty}$ in X, with $x_1 = a$, such that

$$\gamma_{\Phi}(x_n) - n^{-1} < \Phi(x_n, x_{n+1})$$

for all $n \in \mathbb{N}$. Moreover, we can also note that

$$\Phi(x_n, x_{n+1}) \le \sup_{y \ge x_n} \Phi(x_n, y) = \gamma_{\Phi}(x_n)$$

for all $n \in \mathbb{N}$. Therefore, we actually have

$$\gamma_{\Phi}(x_n) - n^{-1} < \Phi(x_n, x_{n+1}) \le \gamma_{\Phi}(x_n)$$

for all $n \in \mathbb{N}$. Hence, by using the monotonicity of the sequence $\big(\gamma_{\Phi}(x_n)\big)_{n=1}^{\infty}$ and some basic theorems on the limits of sequences in $\overline{\mathbb{R}}$, we can infer that

$$\lim_{n\to\infty} \gamma_{\Phi}(x_n) = \lim_{n\to\infty} \Phi(x_n, x_{n+1}).$$

$\square$

Now, by using the above lemma, we can easily prove the following generalized ordering principle.

Theorem 3. *If* Φ *is as in* Lemma 1 *and* $\alpha \in \overline{\mathbb{R}}$ *such that*

(4) each increasing sequence $(x_n)_{n=1}^{\infty}$ *in* X, *with* $x_1 = a$ *is bounded above and satisfies*

$$\underline{\lim}_{\,n\to\infty}\, \Phi(x_n, x_{n+1}) \le \alpha\,;$$

then there exists $b \in X$, *with* $a \le b$, *such that* $\gamma_\Phi(b) \le \alpha$.

Proof. If $(x_n)_{n=1}^{\infty}$ is as *Lemma 1*, then by (4) we have

$$\lim_{n\to\infty} \gamma_\Phi(x_n) = \lim_{n\to\infty} \Phi(x_n, x_{n+1}) = \underline{\lim}_{\,n\to\infty}\, \Phi(x_n, x_{n+1}) \le \alpha\,.$$

Moreover, by (4), there exists $b \in X$ such that $x_n \le b$ for all $n \in \mathbb{N}$. Thus, in particular $a = x_1 \le b$. Moreover, by (1) it is clear that $\gamma_\Phi(b) \le \gamma_\Phi(x_n)$ for all $n \in \mathbb{N}$, and thus

$$\gamma_\Phi(b) \le \lim_{n\to\infty} \gamma_\Phi(x_n) \le \alpha\,.$$

□

3. Applications of the generalized ordering principle

Theorem 3 easily yields the following extension of the main ordering principle of our former paper [13].

Theorem 4. *Assume that* Φ *is an écart on a goset* X *such that* γ_Φ *is decreasing. Moreover, assume that there exists* $\alpha \in \overline{\mathbb{R}}$ *such that*

(a) $\alpha < \gamma_\Phi(x)$ *for all* $x \in X$;

(b) each increasing sequence $(x_n)_{n=1}^{\infty}$ *in* X, *with* $\sup\limits_{x_n \ge x_1} \Phi(x_1, x_n) < +\infty$, *is bounded above and satisfies* $\underline{\lim}_{\,n\to\infty}\, \Phi(x_n, x_{n+1}) \le \alpha$.

Then, we have $\gamma_\Phi(x) = +\infty$ *for all* $x \in X$.

Proof. If the required assertion is not true, then there exists $a \in X$ such that $\gamma_\Phi(a) < +\infty$. Hence, it is clear that for any sequence $(x_n)_{n=1}^{\infty}$ in X, with $x_1 = a$, we have

$$\sup_{x_n \ge x_1} \Phi(x_1, x_n) \le \sup_{y \ge x_1} \Phi(x_1, y) = \gamma_\Phi(x_1) = \gamma_\Phi(a) < +\infty\,.$$

Therefore, by condition (b) and *Theorem 3*, there exists $b \in X$ such that $\gamma_\Phi(b) \le \alpha$. Moreover, by condition (a), we have $\alpha < \gamma_\Phi(b)$. This contradiction proves the required assertion. □

By using *Theorem 3*, we can also easily establish an extension of the main maximum principle of our former paper [13]. For this, it seems convenient to introduce the following

Definition 4. *An écart* Φ *on a goset* X, *satisfying (1) – (3), will be called admissible at the point* a *if there exists* $\alpha \in \overline{\mathbb{R}}$ *such that, in addition to (4), we also have*

(5) $\alpha < \Phi(x, y)$ *for all* $x, y \in X$ *with* $x < y$ *.*

Now, by calling an element x of a goset X maximal if $x \leq y$ implies $x = y$ for all $y \in X$, we can easily state and prove the following generalized maximum principle.

Theorem 5. *If* X *is a goset and* $a \in X$ *such that there exists an écart* Φ *on* X *which is admissible at* a*, then there exists a maximal element* b *of* X *with* $a \leq b$*.*

Proof. By *Definition 4*, there exists $\alpha \in \overline{\mathbb{R}}$ such that, in addition to $(1)-(3)$, we also have (4) and (5). Thus, in particular by *Theorem 3* there exists $b \in X$, with $a \leq b$, such that $\gamma_\Phi(b) \leq \alpha$, and thus $\Phi(b, y) \leq \alpha$ for all $y \in X$ with $b \leq y$.

Now, it remains only to show that b is maximal. For this, note that if this not the case, then there exists $y \in X$, with $b \leq y$, such that $b \neq y$, and thus $b < y$. Then, by the above property of b, we have $\Phi(b, y) \leq \alpha$. Moreover, by condition (5), we also have $\alpha < \Phi(b, y)$. This contradiction proves the maximality of b. □

Remark 4. *By making some obvious modifications in conditions (4) and (5), we can also easily establish the existence of an element* b *of* X*, with* $a \leq b$*, which is quasi-maximal in the sense that* $b \leq y$ *implies* $y \leq b$ *for all* $y \in X$*.*

Note that if the goset X *is reflexive, then every maximal element of* X *is quasi-maximal. While, if the goset* X *is antisymmetric, then the converse statement holds. Therefore, in a reflexive and antisymmetric goset the two notions coincide.*

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